Reconfigurable Low Density Parity Check (LDPC) Code Interleaving for SISO and MIMO OFDM Systems

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Abstract—This contribution\(^1\) derives an algorithm for optimizing the mapping of irregular and systematic Low Density Parity Check (LDPC) code word bits onto Orthogonal Frequency Division Multiplexing (OFDM) carriers in the context of a frequency selective fading channel in both, a Single-Transmit Single-Receive (SISO channel) and Multiple-Transmit Multiple-Receive (MIMO channel) antennas scenario. The absolute values of the frequency domain channel coefficients are assumed to be known and the LDPC code to be given (contrary to existing proposals where the LDPC code is optimized with respect to a given propagation channel). The proposed solution can alternatively be interpreted as an adaptive interleaver (which is inherently available in software defined radio type of system implementations) optimized for a given channel impulse response. In a typical WLAN (IEEE802.11a/n) scenario, the proposed mapping technique improves the system performance by up to approx. 0.7dB compared to a linear (direct) mapping.

I. INTRODUCTION

Low Density Parity Check (LDPC) codes have been extensively studied, originally by [1] and more recently by [2], [3] and others. Based on these results, LDPC codes have been designed outperforming Turbo Codes on Additive White Gaussian Noise (AWGN) channels [4]. While rather precise derivation techniques exist for asymptotic LDPC codes [5] under the pseudonym density evolution, the derivation complexity is important and has been subject to further studies: e.g., [4] proposes an elegant way to deal numerically with the calculation of the distribution of combined random variables (RVs) as they occur in [5]'s algorithm. The same authors introduce a simplified technique based on a Gaussian approximation (message densities are approximated to be Gaussian or Gaussian mixtures for irregular LDPC codes [6]) [7]; they prove that the calculation complexity is reduced by several orders of magnitude while the results often are as accurate as the ones obtained from full density evolution.

In the framework of this paper, we focus on a practical context by considering the efficient use of LDPC codes for Orthogonal Frequency Division Multiplexing (OFDM) based systems in combination with frequency selective fading channels, as it typically occurs in Wireless Local Area Network (WLAN) systems, e.g. IEEE802.11a/n [8], [9]. For this purpose, we build on results and ideas of [10]: LDPC analysis techniques are derived for OFDM systems by approximating the frequency domain channel as a step-profile where each sub-band is characterized by a constant complex channel coefficient. For each sub-band, a sub-LDPC code is defined and considered for the global optimization.

Moreover, a new optimization technique is proposed taking the number of iterations in the LDPC decoding process as well as the resulting error probability of data bits into account. This allows to optimize the allocation of data and redundancy bits which is not possible for density evolution based approaches.

In this paper, we extend the LDPC optimization algorithm of [10] to a sub-carrier based expression (omitting the notion of sub-bands covering several carriers): exploiting that in the context of OFDM the frequency selective fading channel is transformed into a set of parallel attenuations in the discrete frequency domain. Both, the derivation of optimum LDPC codes and the carrier-allocation (which can be interpreted as an adaptive interleaving) for data and redundancy bits are studied. While both optimization steps are ideally performed jointly, the carrier-allocation algorithm is applicable to existing LDPC codes which are not necessarily optimized for a given frequency selective propagation channel. The corresponding results will prove to be applicable to a context where LDPC codes and/or the corresponding carrier mapping are derived based on approximate estimates of the propagation channel. Thanks to the Gaussian approximation, a reliable result is expected (relying on the conclusions of [7]) at low arithmetical calculation complexity requirements.

This paper is organized as follows. Section II defines the notations and assumptions we use. The presentation of a new optimization procedure adapted to the OFDM context follows in section III: two different low-complexity algorithms are derived based on different assumptions on the structure of the LDPC code. Simulation results and a final conclusion are respectively given in sections IV and V.

II. NOTATIONS AND DEFINITIONS

A. OFDM modulation

This section briefly presents the basic definitions introduced in [11], [12] for an \(N\) carrier Cyclic Prefix OFDM (CP-OFDM) system: \(\tilde{s}_n(i)^2\) is first modulated by the IFFT matrix \(F_N^H = \frac{1}{\sqrt{N}} W_N^{iH}, 0 \leq i < N, 0 \leq j < N\ and \ W_N = e^{-j2\pi kN}\). This operation is followed by the insertion of the cyclic extension which is performed by premultiplication of the following expression:

\[
\begin{bmatrix}
0_{D,N-D} & I_D \\
I_N & \end{bmatrix}
\]

\(D\), \(P\)

\(2\)Lower (upper) boldface symbols will be used for column vectors (matrices) sometimes with subscripts \(N\) or \(F\) emphasizing their sizes (for square matrices only); tilde will denote frequency domain quantities; argument \(i\) will be used to index blocks of symbols; \(H\) \((\cdot)^*\) will denote Hermitian (Transpose) and \((\cdot)^*\) is the complex conjugate.
Without loss of generality, the elements of $s_N(i) = F^H\tilde{S}_N(i)$ are assumed to be i.i.d. and zero mean random variables of variance $\sigma^2 = 1$. The samples of are then sent sequentially through the channel, modeled here as an $L$th-order FIR filter $C(z):= \sum_{p=0}^{L} c_p z^{-p}$. The OFDM system is designed such that the postfix duration exceeds the channel memory $L \leq D$.

Let $C_{\text{ISI}}(P)$ and $C_{\text{HI}}(P)$ be respectively the Toeplitz lower and upper triangular matrices of first column $[c_0,c_1,\ldots,c_L,0,\cdots,0]^T$ and first row $[0,\cdots,0,c_L,\ldots,c_1]$. As already explained in [13], after truncation of the cyclic prefix in the receiver, the channel convolution can be expressed as a circular convolution: $r_N(i):=(C_{\text{ISI}}(N)+C_{\text{HI}}(N))s_N(i)+\nu_N(i)$. $C_{\text{ISI}}(N)$ and $C_{\text{HI}}(N)$ represent respectively the intra-symbol and inter-block interference. $\nu_N(i)$ is the $i$th AWGN vector of i.i.d. elements with variance $\sigma^2_p$.

### B. MIMO processing

For application of LDPC interleaving to MIMO systems, we will consider one of the multiple antenna schemes that are included in the 1st draft of IEEE802.11n [14]. This scheme consists in the transmission of two spatial streams over three antennas, as illustrated on Fig. 1, where $\tilde{S}_N(2)(i)$ denotes the $i$th symbol vector of the $k$th stream. As a trade-off between transmit diversity and rate increase, Alamouti Space-Time Block Coding is applied to the 1st stream to be sent on antennas #1 and #2, whereas the uncoded 2nd stream is transmitted on antenna #3. This hybrid scheme also provides the maximum rate increase to the asymmetrical antenna configuration 3Tx 2Rx, which can result from a limited number of antennas on handsets for instance.

### C. LDPC mapping

[10] points out that the choice of the constellation mapping (i.e. the set of $s_N(i)$ amplitudes) plays an important role for the system performance analysis if data bits are encoded based on LDPC Codes. Log-likelihood ratio (LLR) based messages generated from BPSK and QPSK constellations are Gaussian, symmetric and consistent as required for density evolution [3].

Defining $t_v$ as the variable node degree and $t_r$ as the parity check node degree, a regular LDPC code characterized by the parameter set $(Z_{\text{LDPC}},t_v,t_r)$ is a linear block code defined by a sparse parity check matrix $H$ of dimension $Z_{\text{LDPC}} \times M_{\text{LDPC}}$ where $Z_{\text{LDPC}} = \frac{1}{2} M_{\text{LDPC}}$ [1]. The code words $m_{\text{LDPC}}$ consists of $M_{\text{LDPC}}$ bits which all satisfy $Z_{\text{LDPC}}$ parity check equations:

$$H m_{\text{LDPC}} = 0_{Z_{\text{LDPC}}}$$

There are $K_{\text{LDPC}} = M_{\text{LDPC}} - Z_{\text{LDPC}}$ information bits.

A further generalization of LDPC codes has been introduced by [6] with irregular LDPC codes characterized by a non-uniform distribution of ‘1’s over the rows and columns of the $H$ matrix. Typically, the code is represented by two polynomials; $\lambda_i(r_j)$ respectively represent the percentage of branches connected to variable nodes (check nodes) of degree $i$ ($j$). $t_{r_{\text{max}}}$ ($t_{v_{\text{max}}}$) is the maximum connection degree of variable nodes (check nodes):

$$\lambda(x) = \sum_{i=2}^{t_{v_{\text{max}}}} \lambda_i x^{i-1}, \rho(x) = \sum_{j=2}^{t_{r_{\text{max}}}} \rho_j x^{j-1}$$

In the framework of this paper, only systematic LDPC codes are considered. The corresponding generating matrix $G$ of the LDPC code is applied to the initial vector of information bits $b$ as follows: $m = Gb$ with $HG = 0$. Moreover, the analysis is limited to binary LDPC codes (i.e. all elements are in $GF(2)$).

As commonly used, we choose to work with LLR messages in combination with received carrier amplitudes $r$:

$$v = \log\frac{p(r|x=1)}{p(r|x=-1)}$$

are the output messages of variable node where $x$ is the bit value of the corresponding node. Likewise, the output messages of a check node are defined as $u = \log\frac{p(r'|x=1)}{p(r'|x=-1)}$ where $x'$ is the bit value of the variable node arriving from the check node and $r'$ contains all information available to a check node up to the present iteration. Under sum-product decoding [7], $v$ is equal to the sum of the all incoming LLRs: $v = u_0 + \sum_{i=1}^{d_c-1} u_i$ where $u_0$ is the observed LLR of the decoded bit and $u_i, i=1,\cdots,d_c-1$ represent all incoming LLRs from neighbors of the variable node except from the check node that will get the message $v$ (i.e. its input must be independent of its previous output). The $u$ messages are then recalculated based on the so-called “tanh” rule [15]:

$$\tanh\left(\frac{v}{2}\right) = \prod_{j=1}^{d_c-1} \tanh\left(\frac{v_j}{2}\right)$$

where $v_j, j=1,\cdots,d_c-1$ are the incoming LLRs from the $d_c-1$ neighbors of a check node. Note that the message of the node itself ($j=0$) is omitted in the product calculation.

With the notations and definitions presented above, the following sections will show how to derive a suitable LDPC code assuming the channel is known and time-invariant.

### III. LDPC CODE OPTIMIZATION WITH KNOWN AND TIME-ININVARIANT CHANNEL

We choose to perform the optimization of LDPC codes based on the Gaussian Approximation assumption which has been studied and validated in [7] due to its inherent low-complexity implementation properties. However, instead of performing an asymptotic threshold optimization (i.e. a joint search for a matrix $H$ and a minimum SNR level which lead to error free decoding properties if the matrix size and the number of decoding iterations tends towards infinity) we apply the idea of [10] to optimize the error probability of useful data bits with a fixed number of iterations. As [10] mentions correctly, such an approach may be slightly inconsistent, since asymptotic and non-asymptotic properties are mixed up. However, this analysis is better adapted to a practical case requiring a limited number of decoding iterations and [10] shows that the results of the corresponding code optimization have shown satisfying performances for small code word sizes.

In the sequel, it is shown that a generic optimization of the LDPC code-word mapping onto OFDM carriers is difficult to achieve; instead, we propose to apply in typical practical scenarios the following analysis approach: it will be shown that the LDPC code-word should be mapped onto OFDM carriers such that the message values from all variable nodes in the factor...
graph should be identical in the belief propagation decoding. In an asymptotic context (number of OFDM carriers $N \to \infty$, LDPC codeword size $\to \infty$, number of different node degrees $\to \infty$) this is theoretically feasible; note that [10] observed an identical behavior if new LDPC codes are derived for a given propagation channel and in combination with OFDM symbols. It should be noticed, however, that a small number of different node degrees is often defined and this result leads to a sub-optimum optimization result.

Denoting the mean values of $u$ (v) by $m_u$ ($m_v$), the update equation at iteration $l$ becomes for every node with $d_v$ branches [7]:

$$m_v^{(l)} = m_v^{(0)} + (d_v - 1)m_u^{(l-1)}$$

with $m_u^{(0)} = 0$. I.e. the arriving messages to each node are assumed to be identical for the whole graph. For $m_u^{(l)}$, the update equation leads to the following expression [7]:

$$E \left[ \tan \left( \frac{u^{(l)}}{2} \right) \right] = E \left[ \tan \left( \frac{u^{(0)}}{2} \right) \right]^{d_v - 1}$$

In the case of OFDM systems, the initial LLR message depends on the channel coefficient $\tilde{c}_k$ of the corresponding carrier $k$ providing the bit information [10]: $m_{0u} = m_{0u}(\tilde{c}_k)$. Contrary to the sub-band-model in [10], we express the updating equation based on the channel coefficient of each OFDM carrier and the node connection degree (i.e. number of branches) $\zeta(k)$ of each bit modulated on the corresponding sub-carrier. The incoming means $m_u^{(l-1)}$ in (3) are, however, still assumed to be constant for each node. The corresponding updating equation is expressed with the help of $\phi(x)$ (convex for $x > 0$) defined in [7] (see same reference for low complexity approximations of $\phi(x)$):

$$\phi(x) = 1 - \frac{1}{\sqrt{4\pi}} \int \tan \left( \frac{y}{2} \right) e^{-\frac{y^2}{4}} du \text{ for } x > 0 \text{ and } \phi(x) = 0 \text{ otherwise.}$$

Based on these expressions, an iterative expression for $m_u^{(l)}$ as a function of $m_u^{(l-1)}, \zeta(k), m_{0u}(\tilde{c}_k)$ and $k = 0, \ldots, N - 1$ can be derived as presented in [16]. After a maximum number of $l = L$ iterations, the output means corresponding to carrier $k$ are calculated taking all incoming $m_u^{(l)}$ into account:

$$m_{uk} = m_{0u}(\tilde{c}_k) + \zeta(k)m_u^{(L)}$$

Prior to the calculation of the BER, we define:

$$\psi(k) := \begin{cases} 2 & \text{if carrier } k \text{ carries 2 data bits} \\ 1 & \text{if carrier } k \text{ carries 1 data bit} \\ 0 & \text{if carrier } k \text{ carries 2 redundancy bits} \end{cases}$$

The BER is then calculated based on the results of [10] with

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{t^2}{2}} dt: P_e = \frac{1}{N} \sum_{k=0}^{N-1} \psi(k) \frac{1}{\sum_{k=0}^{N-1} \psi(k)} Q \left( \sqrt{m_{0u}(\tilde{c}_k) + \zeta(k)m_u^{(l)}} \right).$$

The optimization of the data bit attribution to OFDM carriers $\psi(x)$ is obtained as follows: $\psi(k)_{opt} = \arg\min_{\psi(k)} P_e(\tilde{r}, \langle \sigma_{uk}, k \rangle, \psi(k))$.

The following section will study the optimization of the LDPC code word mapping onto OFDM carriers with the goal to minimize the decoding error probability.

A. Direct interpretation of the message passing update equation

The goal is to choose the LDPC code-word mapping onto OFDM carriers such that the message passing LDPC decoding algorithm provides optimum performance result as defined above. For this purpose, the channel coefficients of the different OFDM carriers are taken into account as well as the variable/check node degrees of the applied LDPC code.

The optimization is prepared by recalling Jensen’s inequality for convex functions $f(\cdot)$:

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \geq f \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)$$

and by observing that

$$\phi^{-1} \left[ 1 - \frac{1}{2} \left( \phi(m_{0u}(\tilde{c}_k) + \varepsilon) + \phi(m_{0u} - \varepsilon) \right) \right]^{1/\varepsilon} \leq \phi^{-1} \left[ 1 - \phi(m_{0u}(\tilde{c}_k)) \right]^{1/\varepsilon}$$

for $\varepsilon > 0$, $m_{0u}(\tilde{c}_k) > 0$, as shown in [16].

In order to maximize $m_u^{(l)}$, it is required to minimize

$$\sum_{k=0}^{N-1} \psi(k) \left( m_{0u}(\tilde{c}_k) + \zeta(k)m_u^{(l-1)} \right).$$

It is obvious that this goal is achieved if the arguments of $\phi(\cdot)$ are equal for all $k$:

$$\left( m_{0u}(\tilde{c}_k) + \zeta(k)m_u^{(l-1)} \right) = \left( m_{0u}(\tilde{c}_{k'}) + \zeta(k')m_u^{(l-1)} \right), \forall k, k'.$$

Equation (6) guarantees that all messages leaving the variable nodes and arriving at the check nodes in the factor graph are of identical value. If this property is given, the LDPC code-word mapping is optimum (with the limitations inherent to the Gaussian Approximation approach). Note that we assume for the optimization process that the observed messages $m_{0u}(\tilde{c}_k)$ have the correct sign with respect to the corresponding information bit that it carries.

If equality cannot be achieved in (6) (due to the given channel coefficients $\tilde{c}_k$, a limited number of different variable node degrees $\zeta(k) - 1$, etc.), the above mentioned approaches do not identify the optimum solution. In order to mitigate this issue, it is proposed to proceed as follows: i) perform a 2nd (or higher) order approximation of the message passing update equation by choosing the development point as the optimum solution derived above and ii) optimize the approximated expression by evaluation of possible channel coefficient ($\tilde{c}_k$) and variable node degree ($\zeta(k) - 1$) mapping combinations. With $\phi(x) = \phi(x_0 + \frac{\partial \phi}{\partial x_0}(x-x_0)) + \frac{\partial^2 \phi}{\partial x_0^2}(x-x_0)^2 + O(x^3) \approx \phi_0 + \frac{\partial \phi}{\partial x_0}(x-x_0) + \frac{\partial^2 \phi}{\partial x_0^2}(x-x_0)^2$, we obtain:

$$\sum_{k=0}^{N-1} \psi(k) \left( m_{0u}(\tilde{c}_k) + \zeta(k)m_u^{(l-1)} \right) \approx \sum_{k=0}^{N-1} \psi(k) \left( m_{0u}(\tilde{c}_k) + \zeta(k)m_u^{(l-1)} \right).$$

It is then immediately confirmed the intuitive solution that the optimum is achieved by approximating the previously derived mapping result, given by equation (6), as closely as possible.

B. An algorithm for LDPC code word mapping onto OFDM carriers

The optimization of the LDPC code word mapping onto OFDM carriers is based on the following comments with respect to the results derived in the previous sections:

1) $P_e$ (given above) is calculated over the data bits only; redundancy bits are not considered for the final BER.

2) Typically, an LDPC code is larger than one OFDM symbol; still, the definition of $P_e$ given above is applicable if the length of the codeword is an integer multiple of the number of bits modulated in one OFDM symbol.
3) Both, the resulting degree distributions $\lambda(x)$, $\rho(x)$ and the data bit attribution to OFDM carriers depends on the channel coefficients $c$.

4) Ideally, a joint optimization of $\lambda(x)$, $\rho(x)$ and the data bit attribution to OFDM carriers is performed [10].

If only an optimization concerning the data bit attribution to OFDM carriers is required for a given LDPC code, the upper result can be interpreted in a first approximation as follows: In order to minimize the BER, the expression $m_{\text{in}}(\hat{c}_k) + \zeta(k)m_{\text{rd}}(L)$ in the $P_z$ expression (see above) must be as large as possible. Thus,

1) since $P_z$ is calculated only over useful data bits, use lowest order nodes (i.e. $\zeta(k)$ is minimum) for redundancy nodes; i.e. the initial channel observations $m_{\text{in}}(\hat{c}_k)$ are used in order to maximize expression (6).

2) moreover, put data bits on the strongest carriers (i.e. $|c_k|$ is maximum) in order to maximize $m_{\text{in}}(\hat{c}_k)$.

3) assure a maximum final $m_{\text{rd}}(L)$; with Jensen's theorem, the argument of $\phi(\cdot)$ in (4) needs to be as homogeneous as possible;

4) if homogeneous arguments of $\phi(\cdot)$ in (4) cannot be achieved in all cases, try to maximize the arguments for these exceptions since $\phi(x \to \infty) \to 0$.

This observation is illustrated in Fig.2 for an exemplary OFDM channel and an imaginary LDPC code of rate $R = 1/2$ with variable node degrees 2, 3 and 4. It is interesting to note that

4) Use the carriers with the weakest attenuation for data bits and perform an identical mapping as for redundancy bits.

The performance of a corresponding optimization in the context of a typical propagation channel and a typical LDPC code is presented in the following.

For MIMO systems based on the hybrid SDM/STBC scheme presented in Fig.1, the mapping algorithm can be extended as follows:

1) The Serial to Parallel operation to form two spatial streams is performed such that the information and redundancy bits are allocated to the coded and uncoded streams respectively.

2) The Signal to Noise Ratios (SNRs) observed at the receiver after MIMO equalization are duplicated and regrouped in a new vector $\hat{c}_{\text{rd}}$. The ordering of this vector is made such that the $\frac{\phi(\cdot)}{2}$ smallest elements correspond to the SNRs experienced by the uncoded stream.

3) We then apply the same operations as in the single antenna case.

Note that the 1st step of the mapping algorithm for hybrid SDM/STBC can be implemented in any system as it does not depend on CSI knowledge at the Tx.

IV. SIMULATION RESULTS

Simulation results are presented in the following based on the rate $R = 1/2$ LDPC code proposed in the TGnSync IEEE802.11n draft specification [9]:

Parity check matrices $H$ used in the encoding procedure are derived from one of the “base” parity check matrices, $H_b$, specified below. One base matrix is defined per code rate. Size of a base parity check matrix is denoted as $Z_b \times M_b$. $M_b$, the number of columns in the base matrix, is fixed for all code rates, $M_b = 24$. $Z_b$, the number of rows in the base matrix, depends on the code rate as follows: $Z_b := M_b(1 - R)$. Parity check matrix $H$ of size $Z_{LDP} \times M_{LDP}$ is generated by expanding the base matrix for the selected rate, $H_b$. $z$-times: $z = Z_{LDPC} / Z_b = M_{LDPC} / M_b$. The expansion operation is defined by element values of the base matrix. Each non-negative base matrix element, $s$, is replaced by a $z \times z$ identity matrix, $I_z$, cyclically shifted to the right $s = s \mod (z)$ times. Each negative number $(−1)$ in the base matrix is replaced by a $z \times z$ zero matrix, $O_z$. For the codeword of size 576 bits, $z = 24$, for codeword of size 1152 bits, $z = 48$, and for the codeword of size 1728 bits, $z = 72$. The base matrices specification is for $R = 1/2$ and $Z_b \times M_b = 12 \times 24$ defined as given in equation (7). Simulations are performed over here for the 576 bits LDPC-codewords (100,000 code words are simulated per SNR value combined with the TGn-D channel definitions [17]) applying the mapping optimization approach are presented in Fig.3 for the PER; SISO (i.e., a standard single transmit single receive antenna CP-OFDM chain is used as presented in II-A) and MIMO simulations (i.e., the hybrid SDM/STBC scheme presented in section II-B is used for illustration purposes) one LDPC code-word of 576 bits each is considered to be a packet. It should be pointed out that the PER results are improved by approx. 0.7dB for the SISO case and 0.3dB for the MIMO case in the considered SNR range. Since the main argument for using LDPC codes over standard convolution codes focuses on the inherent gain of approx. 2.5dB in favour of LDPC, the additional gains that are observed by the optimized mapping are well motivated. The reduced gain for the 3x2 MIMO configuration originates in
the fact that the multi-antenna approach reduces fading effects on the received and demodulated carrier constellations (a large number of antennas leads to a quasi-AWGN behaviour) and the impact of the interleaving is thus limited. As noted in section III-B, if homogeneous arguments of $\phi(\cdot)$ in (4) cannot be achieved in all cases, try to maximize the arguments for these exceptions (since $\phi(x \to \infty) \to 0$).


(7)

VI. ACKNOWLEDGMENT

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